Hochschild Cohomology of skew group rings and invariants *

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In memory of Sheila Brenner

Abstract

Let A be a k-algebra and G be a group acting on A. We show that G also acts on the Hochschild cohomology algebra $HH^\bullet(A)$ and that there is a monomorphism of rings $HH^\bullet(A)^G \hookrightarrow HH^\bullet(A[G])$. That allows us to show the existence of a monomorphism from $HH^\bullet(\widetilde{A})^G$ into $HH^\bullet(A)$, where $\widetilde{A}$ is a Galois covering with group G.

1 Introduction

The Hochschild cohomology groups were introduced by Hochschild fifty years ago, but they have been investigated lately under different aspects by many authors.

In this work our interest is to study the Hochschild cohomology of skew group rings and of certain Koszul algebras. Our main purpose is comparing the Hochschild cohomology algebra $HH^\bullet(A) = \oplus_{i\geq 0}\operatorname{Ext}^i_{A^e}(A,A)$ of a k-algebra A with the Hochschild cohomology algebra $HH^\bullet(A[G])$ of the skew group algebra $A[G]$, with G being a

finite group acting on A. We obtain the following relation between these cohomology algebras.

Theorem 2.9. *Let A be a k-algebra and G be a finite group acting on A. Then G acts on the Hochschild cohomology k-algebra $HH^\bullet(A)$, and there is a ring monomorphism: $HH^\bullet(A)^G \hookrightarrow HH^\bullet(A[G])$.*

For finite groups, it is known that there is a strong connection between skew group rings and Galois coverings, and between skew group rings and smash products of graded algebras (see [6]). These facts and the existence of the monomorphism in Theorem above lead us to investigate a possible relation between the Hochschild cohomology algebras of G-graded algebras and their covering algebras defined by G. In this direction, let A be a G-graded algebra, with G being a finite group. We consider the covering algebra of A defined by G and we relate it with the smash product $A\#kG^*$. Then, as a consequence of Theorem 2.9 and using Theorem of duality coactions, we obtain that G is a group of automorphism of $\widetilde{A}$ and the existence of a ring monomorhism from $HH^\bullet(\widetilde{A})^G$ into $HH^\bullet(A)$.

While we were reviewing the final version of this article we received from C. Cibils and M.J. Redondo a pre-print entiltled *Cartan-Leray spectral sequence for Galois coverings of categories*, [5]. In this pre-print they gave a spectral sequence involving the Hochschild cohomology of Galois coverings and they show that the monomorphism obtained by us is an isomorphism in case the characteristic of the field is zero.

In this work we also study the Hochschild cohomology groups of Koszul algebras of finite global dimension. With this we reach the Hochschild cohomology groups of $\mathbb{C}$-preprojective algebras associated to Euclidean diagrams and of Auslander algebra of standard algebras, since these algebras are examples of Koszul algebras of global dimension two. We obtain a lower bound for dimension of $HH^n(A)$, for A be a Koszul algebra of global dimension n. So as consequence we get that the second Hochschild cohomology group of $\mathbb{C}$-preprojective algebra of Euclidean type does not vanish; and it is also true for Auslander algebra of standard algebras having non projective indecomposable modules isomorphic to their own Auslander-Reiten translate. So, in both these cases the algebras are not rigid (see [9])

We now describe the contents of each section in the paper. In section 2, after recalling some notions and known facts needed along the work, we state and prove the main result of the section - Theorem 2.9. This theorem states that if a group G is a group acting on an algebra A, then G also acts on the Hochschild cohomology algebra of A and there is a ring monomorphism between the fixed points of the Hochschild cohomology algebra of A and the Hochschild cohomology algebra of the skew group ring $A[G]$.

In section 3 we define the covering algebra $\widetilde{A}$ associated to a G-graded algebra

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A, where G is finite group. We also recall the notion of smash product, and we show that this product is isomorphic to that covering algebra. This isomorphism together with Theorem 2.9 and duality coactions gives us a similar relationship between the invariants of Hochschild cohomology ring $HH^\bullet(\widetilde{A})$ and the Hochschild cohomology ring of A.

In section 4 we deal with quadratic algebras. We construct the Koszul complex for quotient of path algebras by quadratic ideals through a similar procedure used by Berger in [2]. The Koszul complex (named bimodule Koszul complex by him) was constructed in [2] for quotient of free associative algebras A and it is minimal graded resolution of A as $A - A$ bimodule, in case A is a generalized Koszul algebra (also called d-Koszul algebras). But it can be also constructed for algebras which are quotient of quiver algebras by ideals generated by elements of degree $d \geq 2$ (see [11]). Our construction here follows closely the one in [11] for Koszul algebras (that is, 2-Koszul). It enables us to obtain a lower bound for the dimension of the n-Hochschild cohomology group of Koszul algebras of global dimensional n, and in consequence the property mentioned above for preprojective algebras of Euclidean-type and Auslander algebra of an algebra A standard.

2   Hochschild cohomology rings and invariants

Given a ring A we denote by A^{op} its opposite ring. For $a \in A$ we denote by $a^o \in A^{op}$ the corresponding element in A^{op}. In case that A is an algebra over a field k we will denote by A^e its enveloping algebra $A \otimes_k A^{op}$. Moreover, if A and B are algebras over k, the algebra $A \otimes_k B$ will be denoted simply by $A \otimes B$. Sometimes, by simplicity, we will not explicit the ground ring of tensor product when it is clear in the context.

We also remark that the category of left modules over the algebra A^e is canonically isomorphic to the category of $A - A$ bimodules. So we use this isomorphism as identification.

Now we recall some definitions and basic facts.

DEFINITION 2.1 *Let A be a ring and G a group. We say that G acts on A if there is a group homomorphism between G and the group $\mathrm{Aut}(A)$ of ring automorphism of A. If this group homomorphism is injective, we say that G acts faithfully on A or that G is a group of automorphism of A.*

We remark that if G acts on A, then G naturally also acts on the opposite ring A^{op}. In case that A is a k-algebra, we will assume that the group $\mathrm{Aut}(A)$ is the group of automorphisms of k-algebras. Moreover, if G and H are groups acting

on the k-algebras A and B, respectively, then the group $G \times H$ acts on $A \otimes_k B$, and consequently $G \times G$ acts on $A \otimes A^{op}$.

Now we are going to recall the definition of *skew group algebra*.

DEFINITION 2.2 *Let A be a ring and G a finite group acting on A. The elements of the skew group ring $A[G]$ are the same as those of the corresponding group ring. Addition is as usual coordinate-wise, and multiplication is extended by bilinearity from the formula $(ag)(bh) = ag(b)gh$, for a and b in A and g and h in G.*

The following statements will be useful later and their verification are routine.

PROPOSITION 2.3 *Let A be a k-algebra and G be a finite group acting on A. Then,*

    i) The rings $(A[G])^{op}$ and $A^{op}[G]$ are isomorphic, via the map $\theta((ag)^o) = (g^{-1}(a))^o g^{-1}$.

    ii) $A[G] \otimes_k (A[G])^{op}$ is isomorphic to $(A \otimes_k A^{op})[G \times G]$, via the map $\psi(ag \otimes (bh)^o) = a \otimes (h^{-1}(b))^o(g, h^{-1})$.

Now we are going to describe certain approach on the category of the left A-modules and the category of the left $A[G]$-modules, where G is a finite group acting on the algebra A. This approach will be very useful for the next sections.

Denoting by Mod-A the category of the left A-modules, for each $g \in G$, we can associate a functor, denoted by ${}^g()$, on Mod-A. This functor associates to each M in Mod-A the module ${}^gM$ defined as follows: ${}^gM = M$ as an abelian group (or k-vector space, in case A is a k-algebra) and for $a \in A$ and $m \in M$, $a \cdot_g m := g(a)m$. On the morphism, ${}^g()$ is defined as the identity. Observe that the functor ${}^g()$ is clearly an exact functor and is an automorphism of Mod-A. We also observe that it is possible to define in a similar faction an automorphism $()^g$ on the category A-Mod of the right A-modules. Furthermore, analogously we could be consider a functor ${}^g()$ on the category of $A - A$ bimodules, by considering ${}^gM = M$ as an $A - A$ bimodule where it has on the left the structure as above and on the right the original structure of M_A (also in a similar way we could have M^g as a $A - A$ bimodule).

The following facts can be verify easily.

PROPOSITION 2.4 *Let A be a ring and G a group acting on A. If g and h are in G and M is a left A-module, then:*

    i) ${}^g({}^hM) = {}^{hg}M$;

ii) $(M^g)^h = M^{gh}$;

iii) $Ag \cong A^g \cong {}^{\bar{g}^{-1}}A$ *(as $A - A$ bimodules).*

Now we recall some basic facts related to $A[G]$-modules (see [16])

PROPOSITION 2.5 *Let A be a ring and G a finite group acting on A. If M is in Mod-$A[G]$, then the map ${}^g\Psi : M \to {}^gM$ given by ${}^g\Psi(m) = g(m)$ defines an isomorphism of A-modules.*

Proof.   Clearly ${}^g\Psi$ preserves the sum. Let $a \in A$ and $m \in M$. So, we have ${}^g\Psi(am) = g(am) = g(a)g(m) = a \cdot_g g(m) = a \cdot_g {}^g\Psi(m)$, what shows that ${}^g\Psi$ is a homomorphism of A-modules. A similar verification shows that the map $m \to g^{-1}(m)$ is the A-morphism inverse of ${}^g\Psi$. $\square$

PROPOSITION 2.6 *(Lemma 4 in [16]) Let A be a k-algebra and G a finite group acting on A. Let M and N be $A[G]$-modules. Then the following statements hold:*

i) *The abelian group $\mathrm{Hom}_A(M, N)$ is a kG-module, with the action $(g * f)(m) = g(f(g^{-1}(m))$. Denote by $\mathrm{Hom}_A(M, N)^G$ the set of fixed points, then $\mathrm{Hom}_A(M, N)^G = \mathrm{Hom}_{A[G]}(M, N)$;*

ii) *For all $i \geq 1$, there is a natural action of G on $\mathrm{Ext}^i_A(M, N)$ and it verifies that $\mathrm{Ext}^i_{A[G]}(M, N) = \mathrm{Ext}^i_A(M, N)^G$. If $g \in G$, $\xi \in \mathrm{Ext}^i_A(M, N)$ and $\eta \in \mathrm{Ext}^j_A(M, N)$, then $g(\xi\eta) = g(\xi)g(\eta)$.*

Proof.
    i) It is easy and well known (see for instance [16]).
    ii) Let $g \in G$. Since ${}^{g^{-1}}()$ is an exact functor, then a given element $\xi \in \mathrm{Ext}^i_A(M, N)$ is taken to an element ${}^{g^{-1}}\xi \in \mathrm{Ext}^i_A({}^{g^{-1}}M, {}^{g^{-1}}N)$. But, since M and N are $A[G]$-modules, by using the isomorphism ${}^{g^{-1}}\Psi$ and its inverse (see Proposition 2.5) we get an exact sequence, denoted by $g(\xi)$, which is an element in $\mathrm{Ext}^i_A(M, N)$. We note that if two exact sequences are representives of the same element in $\mathrm{Ext}^i_A(M, N)$, then their correspondents under $g(-)$ have the same property. Hence it indicates how to define the action. Proposition 2.4 guarantees that it really defines an action of G on $Ext^i_A(M, N)$.
    The rest of the proof follows from the fact that the functor ${}^{g^{-1}}()$ also preserves the A-projective modules and we leave the details to the reader. $\square$
Note: We note that statement ii) in the last proposition could also be proved by remarking that $A[G]$ is a projective A-module, and applying the functor $\mathrm{Hom}_A(-, N)$

to the projective resolution of M as an $A[G]$-module, and observing that the homology obtained at each step is a kG-module.

Now we are going to recall the definition of Hochschild cohomology. We present it closely to the original approach given by Hochschild and it can be found, for instance, in [1, 4, 13, 20].

DEFINITION 2.7 *Let A be an algebra over a commutative ring R, and ${}_A X_A$ be an A-bimodule. The ith-Hochschild cohomology group of A with coefficients in X, denoted by $HH^i(A, X)$, is the ith cohomology group of the following complex:*

$$0 \to X \xrightarrow{d_0} \operatorname{Hom}_R(A, X) \xrightarrow{d_1} \operatorname{Hom}_R(A^{\otimes 2}, X) \cdots \operatorname{Hom}_R(A^{\otimes n}, X) \xrightarrow{d_n} \operatorname{Hom}_R(A^{\otimes n+1}, X) \cdots$$

where, for $n \geq 1$ and $f \in \operatorname{Hom}_R(A^{\otimes n}, X)$, d_n is defined by

$$
\begin{aligned}
d_n(f)(x_1 \otimes x_2 \ldots \otimes x_{n+1}) &= x_1 f(x_2 \otimes \ldots \otimes x_{n+1}) \\
&+ \sum_{i=1}^{n} (-1)^i f(x_1 \otimes \ldots \otimes x_i x_{i+1} \otimes \ldots \otimes x_{n+1}) \\
&+ (-1)^{n+1} f(x_1 \otimes \ldots \otimes x_n) x_{n+1}
\end{aligned}
$$

and for $x \in X$ and $a \in A$, $(d_0 x)(a) = ax - xa$.

In case that $R = k$ is a field, a different way of approaching to Hochschild cohomology groups is to consider the enveloping algebra $A^e = A \otimes_k A^{op}$. In this case $HH^i(A, X) = Ext^i_{A^e}(A, X)$, for all $i \geq 0$. But our particular interest is the example $X = A$, whose $HH^i(A, A)$ is simply denoted by $HH^i(A)$, for $i \geq 0$. These groups are used to define the *Hochschild cohomology algebra* $HH^\bullet(A) = \oplus_{i \geq 0} HH^i(A) = \oplus_{i \geq 0} \operatorname{Ext}^i_{A^e}(A, A)$ with the multiplication induced by the Yoneda product. In this way $HH^\bullet(A)$ is a $\mathbb{Z}$-graded algebra (see for instance [1, 20]).

The facts which we state next point up how important they are to stablish the main result of this section: to relate the Hochschild cohomology algebras $HH^\bullet(A)$ and $HH^\bullet(A[G])$.

Let A be a k-algebra and G be a finite group acting on A. We have seen, in Proposition 2.3, the enveloping algebra $A[G]^e$ of the skew group algebra $A[G]$ is isomorphic to the k-algebra $A^e[G \times G]$. We shall use this isomorphism together with Proposition 2.6 for describing an action of $G \times G$ on the Hochschild cohomology algebra $HH^\bullet(A[G])$, which respects its $\mathbb{Z}$-grading (meaning the action takes an element in $HH^i(A[G])$ to an element in $HH^i(A[G])$).

With this in mind, we recall that $HH^\bullet(A) = \oplus_{i \geq 0} \operatorname{Ext}^i_{A^e}(A, A)$ and $HH^\bullet(A[G]) = \oplus_{i \geq 0} \operatorname{Ext}^i_{A[G]^e}(A[G], A[G])$. We also recall that $A[G]$ is a $A^e[G \times G]$-module, with

the following " multiplications ": $(x \otimes y^o)(\sum_{g \in G} a_g g) = \sum_{g \in G} x a_g g(y) g$, and $(\sigma, \tau)(\sum_{g \in G} a_g g) = \sum_{g \in G} \sigma(a_g) \sigma g \tau^{-1}$, for x, y, a_g in A, and σ, τ in G. Then from Proposition 2.6, for each $i \geq 0$, follows that $G \times G$ acts on $\mathrm{Ext}^i_{A^e}(A[G], A[G])$ and $\mathrm{Ext}^i_{A[G]^e}(A[G], A[G]) \cong (\mathrm{Ext}^i_{A^e}(A[G], A[G]))^{G \times G}$. So, from the action of $G \times G$ on the grading we get the one wanted on $HH^\bullet(A[G])$.

For obtaining the relationship between the Hochschild cohomology algebras of A and of $A[G]$, now we describe a bit more in details the action considered above (the one considered in Proposition 2.6) just for the cases $i = 0$ and $i = 1$, since for $i \geq 2$ the procedures are analogous to the one $i = 1$.

First we write $A[G] = \coprod_{g \in G} Ag$, as an A^e- module (or $A - A$ bimodule), and so we get $HH^i(A[G]) = \mathrm{Ext}^i_{A[G]^e}(A[G], A[G]) \cong \coprod_{(g,h) \in G \times G} \mathrm{Ext}^i_{A^e}(Ag, Ah)$, for each $i \geq 0$.

1. The action in case $i = 0$

For each g, h in G, let $f_{(g,h)} \in \mathrm{Hom}_{A^e}(Ag, Ah)$ and $(\sigma, \tau) \in G \times G$. Then we consider the element $(\sigma, \tau)(f_{(g,h)}) \in \mathrm{Hom}_{A^e}(A\sigma g \tau^{-1}, A\sigma h \tau^{-1})$ defined by $(\sigma, \tau)(f_{(g,h)})(a\sigma g \tau^{-1}) = a\sigma f(g)\tau^{-1}$, with $a \in A$.

Clearly it defines an action on $\mathrm{Hom}_{A^e}(A[G], A[G]) = \coprod_{(g,h) \in G \times G} \mathrm{Hom}_{A^e}(Ag, Ah)$ and it is the action considered in Proposition 2.6.i).

2. The action in case $i = 1$.

Let $\xi_{(g,h)} : 0 \to Ah \to L \to Ag \to 0$ be a representative of an element in $\mathrm{Ext}^1_{A^e}(Ag, Ah)$ and $(\sigma, \tau) \in G \times G$. By applying the exact functor $^{(\sigma^{-1}, \tau^{-1})}()$ to this exact sequence and using the isomorphism $^{(\sigma^{-1}, \tau^{-1})}Ah \xrightarrow{\sim} A\sigma h \tau^{-1}$ we obtain the element, denoted by $(\sigma, \tau)(\xi_{(g,h)}) \in \mathrm{Ext}^1_{A^e}(A\sigma g \tau^{-1}, A\sigma h \tau^{-1})$. It is easy to verify that it really defines an action on $\coprod_{(g,h) \in G \times G} \mathrm{Ext}^1_{A^e}(Ag, Ah)$, and it is the action mentioned in Proposition 2.6.ii).

Remark 2.8 *We observe, in particular, that the subspace $\coprod_{g \in G} \mathrm{Ext}^i_{A^e}(Ag, Ag)$ of $\mathrm{Ext}^i_{A^e}(A[G], A[G])$ under the action of $G \times G$ is taken to itself.*

On the other hand we also note that, for each $i \geq 0$ and $g \in G$, $\mathrm{Ext}^i_{A^e}(A, A)$ is canonically isomorphic to $\mathrm{Ext}^i_{A^e}(Ag, Ag)$ (see 2.4.iii). So, with this identification, we can consider an action of G on $HH^i(A) = \mathrm{Ext}^i_{A^e}(A, A)$ such as the one given by the following: for each $g \in G$ and $\xi \in HH^i(A)$, $g \cdot \xi := (g, g)(\xi)$. Consequently, we obtain that G acts on $HH^\bullet(A)$.

Now we can show the main result of this section which gives a relation between the Hochschild cohomology algebras of A and $A[G]$.

THEOREM 2.9 *Let A be a k-algebra and G be a finite group acting on A. Then G acts on the Hochschild cohomology k-algebra $HH^\bullet(A)$, and there is a ring monomorphism: $HH^\bullet(A)^G \hookrightarrow HH^\bullet(A[G])$.*

Proof.

First we write $A[G] = \coprod_{g \in G} Ag$. Then we remark again that the action of $G \times G$ on $HH^i(A[G])$ $(i \geq 0)$ enables having $HH^i(A[G]) \cong (\mathrm{Ext}^i_{A^e}(A[G], A[G]))^{G \times G} \cong (\coprod_{(g,h) \in G \times G} \mathrm{Ext}^i_{A^e}(Ag, Ah))^{G \times G}$. So, it suggests us to identifying an element $\xi \in \mathrm{Ext}^i_{A^e}(A[G], A[G])$ with a matrix $\xi = (\xi_{(g,h)})_{g,h}$, with $\xi_{(g,h)} \in \mathrm{Ext}^i_{A^e}(Ag, Ah)$.

Now we also remark that $HH^i(A) = \mathrm{Ext}^i_{A^e}(A, A) \cong \mathrm{Ext}^i_{A^e}(Ag, Ag)$, for any $g \in G$. So, according to Remark 2.8, G acts on $HH^i(A)$, and consequently on $HH^\bullet(A)$, as it was indicated there.

Therefore the morphism we are looking for can be defined as: for each $i \geq 0$, given $\xi \in HH^i(A)^G$ we take the element $\theta^i(\xi) \in \coprod_{(g,h) \in G \times G} \mathrm{Ext}^i_{A^e}(Ag, Ah)$ whose matrix representation $\theta^i = (\theta^i(\xi)_{(g,h)})_{g,h}$ is such that:

$$\theta^i(\xi)_{(g,h)} = \begin{cases} 0 & \text{if } g \neq h \\ \xi & \text{if } g = h. \end{cases}$$

Since $\xi \in (HH^i(A))^G$, then, by construction, $\theta^i(\xi)$ is invariant under the action of $G \times G$, and therefore the map $\theta^i : HH^i(A)^G \to HH^i(A[G])$ is defined; and it is not difficult to verify that $\theta : HH^\bullet(A)^G \to HH^\bullet A[G]$, with $\theta = \oplus_i \theta^i$ is a monomorphism of rings. $\qquad \square$

3   Galois covering and Hochschild cohomology.

In this section we are going to apply the main theorem of the last section (Theorem 2.9) to show that also there is a ring monomorphism from $HH^\bullet(\widetilde{A})^G$ into $HH^\bullet(A)$, where $\widetilde{A}$ is the covering algebra of a G-graded k-algebra A.

In [6] it was proven that, for a k-algebra A graded by a finite group G, the smash product $A\#kG^*$ plays the role for graded rings that the skew group algebra $A[G]$ plays for group actions. So, in order to obtain the new pretended monomorphism we shall use the notion of smash product $A\#(kG)^*$, for showing the existence of an isomorphism between this product and the covering algebra of A defined by G and we apply the duality Theorem 3.5 in [6].

We recall here the definition of covering algebra associated to a graded algebra. This definition was introduced in a preliminary version of [12], and it can be found in [15]. The definition of covering algebra coincides with the one given by Green in [10] for quotient of path algebras of quivers.

DEFINITION 3.1 *Let G be a finite group and $A = \coprod_{g \in G} A_g$ be a G-graded k-algebra, with A_g indicating the k-subspace of the homogeneous elements of degree g. The covering k-algebra associated to A, with respect to the given grading, denoted by $\widetilde{A}$, is defined as follows. As k- vector space*
$\widetilde{A} = \coprod_{(g,h) \in G \times G} \widetilde{A}[g,h]$, *where $\widetilde{A}[g,h] = A_{g^{-1}h}$, and the multiplication is defined in the following way: if $\gamma \in \widetilde{A}[g,h]$ and $\gamma' \in \widetilde{A}[g',h']$. The product is in $\widetilde{A}[g,h']$ and it is defined by:*

$$\gamma\gamma' = \begin{cases} 0 & \text{if } g' \neq h \\ \gamma\gamma' & \text{if } g' = h. \end{cases}$$

Remark 3.2 *We observe that G acts freely on $\widetilde{A}$, where the action of an element $\sigma \in G$ consists in to take an element in $\widetilde{A}[g,h]$ to the same element, but now considered as an element in $\widetilde{A}[\sigma g, \sigma h]$. Moreover the canonical vector space epimorphism $F : \widetilde{A} \to A$ which takes $\widetilde{A}[g,h]$ to $A_{g^{-1}h}$, is such that $F\sigma = F$, for all $\sigma \in G$, and the orbit space is A. So $\widetilde{A}$ is a Galois covering defined by G in the sense of Gabriel and others,([3, 17]).*

Now we review the definition of smash product and some facts related to it (see [6]).

DEFINITION 3.3 *Let G be a finite group and $A = \coprod_{g \in G} A_g$ be a G-graded algebra. Let $k[G]^*$ be the dual algebra of $k[G]$, and its natural k-basis $\{p_g | g \in G\}$; that is , for any $g \in G$ and $x = \sum_{h \in G} a_h h \in k[G]$, $p_g(x) = a_g \in k$, and $p_g p_h = \delta_{g,h} p_h$, where $\delta_{g,h}$ is the Kronecker delta. The smash product, denoted by $A\#kG^*$, is the vector space $A \otimes_k k[G]^*$ with the multiplication given by $(a\#p_g)(b\#p_h) = ab_{gh^{-1}}\#p_h$ (here $a\#p_g$ denotes the element $a \otimes p_g$).*

The next proposition was first proved by Green, Marcos and Solberg in a preliminary version of [12].

PROPOSITION 3.4 *Let G a finite group and $A = \coprod_{g \in G} A_g$ be a G-graded algebra. Then the smash product $A\#kG^*$ and the covering algebra $\widetilde{A}$ of A are isomorphic algebras.*

Proof.   Any $a \in A$ can be written uniquely as $a = \sum_{h \in G} a_h$, where $a_h \in A_h$. So we can define the following map $\Psi : A\#k[G]^* \longrightarrow \widetilde{A}$ by

$$\Psi((\sum_{h \in G} a_h)\#p_g) = \sum_{h \in G} a_h \in \coprod_{h \in G} \tilde{A}[g^{-1}h^{-1}, g^{-1}].$$

It is not hard to show that Ψ is a bijective homomorphism of algebras. $\qquad \square$

In Remark 3.2, we have seen the group of grading of A acts on the covering algebra $\widetilde{A}$. Now we also note that using the isomorphism in Proposition 3.4 we get a corresponding action on the smash product $A\#kG^*$, which is given by $g(a\#p_h) = a\#p_{hg}$; and it coincides with the one defined in Lemma 3.3 in [6].

With these remarks, as a consequence of Theorem 2.9, of the isomorphism above and the duality coactions of Cohen-Montgomery (Th.3.5 in [6]) we obtain the following proposition.

PROPOSITION 3.5 *Let G be a finite group and A be a G-graded k-algebra. Let $\widetilde{A}$ be the covering algebra defined by the grading. Then G acts on $HH^\bullet(\widetilde{A})$ and there is a ring monomorphism from $(HH^\bullet(\widetilde{A}))^G$ into $HH^\bullet(A)$*

Proof. As we have seen, in Remark 3.2, G acts on $\widetilde{A}$ as a group of automorphisms. Then, on the one side, from Theorem 2.9 it follows that G also acts on $HH^\bullet(\widetilde{A})$ and there is a monomorphism from from $HH^\bullet(\tilde{A})^G$ to $HH^\bullet((\widetilde{A})[G])$. But, on the other side, by Proposition 3.4, $\tilde{A}$ and $A\#kG^*$ are isomorphic, and according to our remark above this isomorphism leads G to act on the smash product $A\#G$. So, by applying the duality theorem for coactions (Th. 3.5 in [6]), we get that $(A\#kG^*)[G]$ is isomorphic to the matrix ring $M_{|G|}(A)$ where $|G|$ denotes the order of the group G. Since Hochschild cohomology is an invariant by Morita equivalence (in reality is an invariant of derived equivalence, [19, 20]), then it follows that $HH^\bullet(\widetilde{A}[G])$ is isomorphic to $HH^\bullet(A)$, and the proposition is proved. $\qquad \square$

4   The Hochschild cohomology of Koszul algebras

In this section we discuss some facts related to the Hochschild cohomology of Koszul algebras. In particular the ones concerning to the preprojective algebras of Euclidean-type and to Auslander algebras of standard algebras, which are Koszul algebras.

In order to study the Hochschild cohomology of Koszul algebras of finite global dimension we introduce the construction of Koszul complex for quadratic algebras. We are using a similar procedure as was done in [2] and [11] for generalized Koszul algebras (or d- Koszul algebras). According to our comments in the introduction of this article, we use it for 2-Koszul algebra or simply Koszul algebras. So we review some definitions and fix some notations.

Let k be a field and Q be a finite quiver Q. We denote by kQ the path algebra of Q and we indicate by kQ_0 the k-subalgebra whose underlying vector space is the subspace generated by the vertex set Q_0 of Q. If Q_i is the set of paths of length i, then we denotes kQ_i the subspace of kQ generated by Q_i. It is worth to

note that this subspace is a kQ_0 bimodule. In this way, we will consider the path algebra $kQ = \oplus_{i \geq 0} kQ_i$ as a graded algebra with the grading given by the length of the paths.

Let $A = kQ/I$ where Q is a finite quiver and I is a two side ideal of kQ generated by a set of quadratic relations (the k-algebra A is called a quadratic algebra). Denoting by R the set of homogeneous elements of degree two in I that it is viewed as a kQ_0 sub-bimodule contained in kQ_2.

For each $n \geq 2$, we define $K_n = \bigcap_{r+s+2=n} kQ_r.R.kQ_s$. Now we consider the following A^e-modules: $Q^i = 0$, if $i < 0$; $Q^0 = A \otimes_{kQ_0} A$; $Q^1 = A \otimes_{kQ_0} kQ_1 \otimes_{kQ_0} A$, and, for $n \geq 2$, $Q^n = A \otimes_{kQ_0} K_n \otimes_{kQ_0} A$.

It is clear that each Q^i is an A^e-module finitely generated and projective. Observe also that each Q^i is a submodule of $A \otimes_{kQ_0} kQ_i \otimes_{kQ_0} A$, since K_i is contained in kQ_i, for $i \geq 2$.

Now we construct, for each $n \geq 2$, the following A^e-morphisms $f_n : A \otimes_{kQ_0} kQ_n \otimes_{kQ_0} A \to A \otimes_{kQ_0} kQ_{n-1} \otimes_{kQ_0} A$ given by the formula

$$f_n(a \otimes \alpha_1 \cdots \alpha_n \otimes b) = a\alpha_1 \otimes \alpha_2 \cdots \alpha_n \otimes b + (-1)^n a \otimes \alpha_1 \cdots \alpha_{n-1} \otimes \alpha_n b.$$

DEFINITION 4.1 *Let $A = kQ/I$ and Q^i be the A^e-modules as above. Let $d_i : Q^i \to Q^{i-1}$ be the maps such that $d_i = O$, for $i \leq 0$; $d_1(1 \otimes_{kQ_0} \alpha \otimes_{kQ_0} 1) = \alpha \otimes_{kQ_0} 1 - 1 \otimes_{kQ_0} \alpha$, with $\alpha \in Q_1$, and, for $n \geq 2$, d_n is the restriction of f_n to Q^n. It is very easy to see that $d \circ d = 0$. Then the complex $K^*(A) = ((Q^i)_i, (d_i)_i)$ is called the Koszul complex of A.*

With the definition of Koszul complex on the hands, we can utilize it for characterizing the Koszul algebras. In order to get it we take the *augmented Koszul complex of A*:

$\cdots \to Q^n \xrightarrow{d_n} Q^{n-1} \xrightarrow{d_{n-1}} \cdots \to Q^1 \xrightarrow{d_1} Q^0 \xrightarrow{d'_0} A \to 0$ where d_n, $n > 0$, is as in $K^*(A)$ and $d'_0(a \otimes_{kQ_0} b) = ab$

THEOREM 4.2 *[2, 11] Let $A = kQ/I$ be a quadratic algebra. The augmented Koszul complex $\cdots \to Q^n \xrightarrow{d_n} Q^{n-1} \xrightarrow{d_{n-1}} \cdots \to Q^1 \xrightarrow{d_1} Q^0 \xrightarrow{d'_0} A \to 0$ is exact if and only if A is a Koszul algebra.*

In case $A = kQ/I$ is a Koszul algebra, the augmented Koszul complex of A is a minimal graded projective resolution of A as A^e-module. Furthermore, if I is an admissible ideal, then this resolution is also a minimal projective resolution of A in A^e-mod.

The augmented Koszul complex can be used to determine a lower bound for dimension of $HH^n(A)$, when A is a Koszul algebra of global dimension n. It is obtained in the corollary below.

COROLLARY 4.3 *Let $A = KQ/I$ be a Koszul algebra of global dimension n. For each vertex $v \in Q_0$, e_v denotes the associated idempotent of A. Then*

$$\dim_k(HH^n(A)) \geq \dim_k(\coprod_{v \in Q_0} (e_v K_n e_v))$$

.

Proof. Since A is a Koszul algebra and has the global dimension equal n, by Theorem 4.2 we have, using the notations fixed above, that the long exact sequence $0 \to Q^n \overset{d_n}{\to} Q^{n-1} \cdots \to Q^2 \overset{d_2}{\to} Q^1 \overset{d_1}{\to} Q^0 \overset{d'_o}{\to} A \to 0$ is a graded projective resolution of A in A^e-mod. Then the Hochschild cohomology of A can be computed as the cohomology groups of the complex:

$$0 \to \operatorname{Hom}_{A^e}(A \otimes_{A_0} A, A) \overset{d_1^*}{\to} \operatorname{Hom}_{A^e}(A \otimes_{A_0} kQ_1 \otimes_{A_0} A, A) \to \cdots \to$$

$$\operatorname{Hom}_{A^e}(A \otimes_{A_0} K_{n-1} \otimes_{A_0} A, A) \overset{d_n^*}{\to} \operatorname{Hom}_{A^e}(A \otimes_{A_0} K_n \otimes_{A_0} A, A) \to 0 \cdots$$

where $A_0 = kQ_0$.

On the other hand, it is easy to verify that $\operatorname{Hom}_{A^e}(A \otimes_{A_0} A, A) \cong \operatorname{Hom}_{A_0^e}(A_0, A) \cong \coprod_{v \in Q_0} e_v A e_v$, $\operatorname{Hom}_{A^e}(A \otimes_{A_0} kQ_1 \otimes_{A_0} A, A) \cong \operatorname{Hom}_{A_0^e}(kQ_1, A)$, and, for $j \geq 2$, $\operatorname{Hom}_{A^e}(A \otimes_{A_0} K_j \otimes_{A_0} A, A) \cong \operatorname{Hom}_{A_0^e}(K_j, A)$. Hence the last complex is isomorphic to the following one:

$$0 \to \coprod_{v \in Q_0} e_v A e_v \overset{d_1^*}{\to} \operatorname{Hom}_{A_0^e}(kQ_1, A) \cdots \operatorname{Hom}_{A_0^e}(K_{n-1}, A) \overset{d_n^*}{\to} \operatorname{Hom}_{(A_0)^e}(K_n, A) \to 0,$$

where we also are denoting by d_i^* the induced maps by the isomorphisms mentioned above.

We observe that the vector space $\operatorname{Hom}_{(A_0)^e}(K_j, A)$, for $j \geq 2$, can be naturally graded by the induced grading of A; that is, we say that a map f is homogeneous of degree t if its image is contained in homogeneous component of degree t of A. So, it easy to see that the last complex is a complex of graded vector spaces and that the image of each d_j^* is contained in the direct sum of the homogeneous subspaces of degree bigger than zero. Then, for $j \geq 2$, the image of d_j^* does not intersect the degree zero component $(\operatorname{Hom}_{A_0^e}(K_j, A))_0 \cong \coprod_{v \in Q_0}(e_v K_j e_v)$ of $\operatorname{Hom}_{A_0^e}(K_j, A)$. In particular, the degree zero component $(\operatorname{Hom}_{A_0^e}(K_n, A))_0 \cong \coprod_{v \in Q_0}(e_v K_n e_v)$ does not intersect the image of d_n^* and since $HH^n(A) = \operatorname{Coker} d_n^*$, a simple computation of dimensions shows our statement. □

Among the Koszul algebras we are going to point out the $\mathbb{C}$-preprojective algebras of Euclidean type and the Auslander algebra of a standard, representation

finite-type k- algebra. We will see that these algebras are Koszul algebras and as consequence of it, via Corollary 4.3, we get interesting datum for $HH^2(A)$.

First let us review the definition of preprojective algebras.

DEFINITION 4.4 *Let Q be a finite quiver and k a field. Consider the quiver $\hat{Q}$ whose vertex set $\hat{Q}_0 = Q_0$ and the arrows set $\hat{Q}_1 = Q_1 \cup Q_1^{op}$, where Q^{op} denotes the opposite quiver of Q. For each arrow $\alpha \in Q_1$ we write $\hat{\alpha}$ for the corresponding arrow in the opposite quiver. The preprojective k-algebra associated to Q (or briefly the preprojective k-algebra of Q), denoted by $\mathcal{P}(Q)$, is is the k-algebra $k\hat{Q}/I$, where I is the ideal generated by the relations $\sum\limits_{\alpha \in Q_1} \alpha\hat{\alpha}$ and $\sum\limits_{\alpha \in Q_1} \alpha\hat{\alpha}$.*

We remark that it is well-known that the preprojective algebra constructed as above only depends on the underlying graph of the quiver Q; that is, quivers having the same underlying graph define isomorphic preprojective algebras.

The Hochschild cohomology of preprojective algebras associated to Dynkin diagrams A_n were studied in [8].

We mention the following result about preprojective $\mathbb{C}$-algebras associated to Euclidean diagrams (see [7, 14, 18]).

THEOREM 4.5 *The preprojective $\mathbb{C}$-algebras associated to an Euclidean diagram are Morita equivalent to the skew group algebras $\mathbb{C}[x,y][G]$, with G a polyhedral group.*

So this theorem can be used in order to study some properties of the preprojective $\mathbb{C}$-algebra of Euclidean-type through properties of certain skew group rings. For instance, from this theorem we obtain that a preprojective $\mathbb{C}$-algebra associated to Euclidean diagrams have global dimension 2 (recall that $gldim(\mathbb{C}[x,y][G]) = gldim(\mathbb{C}[x,y]) = 2$). Moreover, since the preprojective algebras are always quadratic algebras, then we also get that preprojective $\mathbb{C}$-algebras of Euclidean-type are Koszul algebras.

In this point of view we obtain as a consequence of Corollary 4.3, the following fact about the second Hochschild cohomology group of preprojective $\mathbb{C}$-algebras of Euclidean-type.

COROLLARY 4.6 *Let A be a preprojective $\mathbb{C}$-algebra associated to an Euclidean diagram. Then $HH^2(A) \neq 0$*

Proof. We have that $A = \mathbb{C}\hat{Q}/I$ where Q is an Euclidean diagram. As we have commented above A is a Koszul algebra of global dimension two. Since $e_v K_2 e_v$ is not zero, for any vertex v, the statement follows from the corollary 4.3. □

We note that Theorem 4.5 can be used this to describe the centre of preprojective $\mathbb{C}$-algebras of Euclidean-type, once their centres are the ones of the skew group rings $\mathbb{C}[x, y][G]$, for suitable groups G. Then it seems interesting to study the Hochschild cohomology of the skew group ring, in particular its centre. In order for studying it the following lemma will be useful.

LEMMA 4.7 *Let R be a commutative ring and G be a group acting on R. Let g be an element in G and suppose that there is $\alpha \in R$ such that $\alpha - g(\alpha)$ is not a zero divisor in R. Then $\mathrm{Hom}_{R^e}(R, Rg) = 0$.*

Proof.   Let $f \in \mathrm{Hom}_{R^e}(R, Rg)$. Then $\alpha f(1) = f(1).\alpha = f(1)g(\alpha) = g(\alpha)f(1)$, and it implies that $(\alpha - g(\alpha))f(1) = 0$. Since $\alpha - g(\alpha)$ is not a zero divisor in R, it follows that $f(1) = 0$, and consequently $f = 0$.   □

Now we be able to describe the center of a skew group ring.

PROPOSITION 4.8 *Let R be a commutative domain and G be a finite group of automorphism of R. Then $\mathrm{Center}(R[G])$ is isomorphic to R^G.*

Proof.

First we observe that $Hom_{R^e}(Rg, Rh) \cong \mathrm{Hom}_{R^e}(R, Rhg^{-1})$, for any g, h in G. So, since G acts faithfully on R and R is a domain, by Lemma 4.7 we obtain that $\mathrm{Hom}_{R^e}(Rg, Rh) = 0$, for any $g \neq h$ in G.

Now, if we write $R[G] = \coprod_{g \in G} Rg$ as R^e-module, then we have that $\mathrm{Center}(R[G]) = HH^0(R[G]) = \mathrm{Hom}_{R[G]^e}(R[G], R[G]) = (\mathrm{Hom}_{R^e}(R[G], R[G]))^{G \times G} = (\coprod_{(g,h) \in G \times G} Hom_{R^e}(Rg, Rh))^{G \times G} \cong (\coprod_{g \in G} Hom_{R^e}(Rg, Rg))^{G \times G}$.

Recalling the action defined in section 2 and Remark 2.8 we have that $(\coprod_{g \in G} Hom_{R^e}(Rg, Rg))^{G \times G} \cong (Hom_{R^e}(R, R))^G \cong R^G$, and the statement is proved.

□

We remark that the proof of Proposition 4.8 could be obtained by a direct computation, but we have optioned by the proof above for illustrating how to use our methods.

For the next corollary we need some additional terminology. We are going to consider the Auslander algebra associated to a k-algebra of representation finite type. So, we recall the definition of Auslander algebras.

Let A be a k-algebra of representation finite type (i.e. up to isomorphism there exist only finitely many indecomposable A-modules). Let $X_1, X_2, \cdots X_m$ be a list of representatives from the isomorphism classes of indecomposable A-modules and let $X = \oplus_i X_i$. The k-algebra $\Lambda = \mathrm{End}_A(X)$ is called *Auslander algebra of A*. Recall that A is said to be *standard* if Λ is isomorphic to the quotient of the path

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algebra $k\Gamma_A$ of the Auslander-Reiten quiver Γ_A of A by the ideal generated by the mesh relations.

We denote by mod-A the category of finitely generated left A-modules, by ind A the subcategory of mod-A with one representative of each isoclass of indecomposable A-module and by τ_A the Auslander-Reiten translate DTr.

COROLLARY 4.9 *Let A be a standard representation-finite type k-algebra and Λ be its Auslander algebra. Then* $\dim_k HH^2(\Lambda) \geq \#\{M \in \mathrm{ind}\, A : \tau_A M = M\}$

Proof. It is known that the Auslander algebra Λ of a representation-finite algebra A has global dimension two. Moreover, since A is standard, we have that $\Lambda \cong k\Gamma_A/I$, where Γ_A is the Auslander-Reiten quiver of A and I is the ideal generated by the mesh relations (so quadratic relations). Hence Γ is a Koszul algebra.

By construction of the quiver Γ_A and by the conditions on I, it is clear that the number of elements of the set $\{M \in \mathrm{ind}\, A : \tau_A M) = M\}$ is the dimension of the degree zero component of $Hom_{k(\Gamma_A)_0^e}(K_2, \Lambda)$. But that component is isomorphic to $\coprod_{v \in (\Gamma_A)_0} e_v K_2 e_v$, and therefore the result follows from corollary 4.3 $\qquad\square$

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